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FREE-SURFACE FLOW OVER CURVED SURFACES PART I: PERTURBATION ANALYSIS

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SUMMARY

The standard two-dimensional shallow water equation formulation assumes a mild bed slope and no curvature effect. These assumptions limit the applicability of these equations for some important classes of problems. In particular, flow over a spillway is affected by the bed curvature via a decidedly non-hydrostatic pressure distribution. A detailed derivation of a more general equation set is given here in Part I. The method relies upon a perturbation expansion to simplify a bed-fitted co-ordinate configuration of the three-dimensional Euler equations. The resulting equations are essentially the equivalent of the two-dimensional shallow water equations but with curvature included and without the mild slope assumption. A finite element analysis and flume result are given in Part II. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: curved surface; shallow water; non-hydrostatic; spillway; finite element

1. DISCUSSION

The standard shallow water equations are hydrostatic and assume a small channel slope with no curvature [1]. Each of these assumptions limits their use in certain applications, such as spillway simulations. Dressler [2] derived a more general set of one-dimensional shallow water equations in which channel bed curvature was included. This treatment followed the procedure originated by Friedrichs [3] and extended by Keller [4] (see also Stoker [1]) by utilizing an asymptotic expansion in a shallowness parameter $\varepsilon = (h/l)^2$ where l is the radius of curvature of the free surface and h is the depth. This formulation leads to the equations

$$\hat{u}_t + \frac{\hat{u}\hat{u}_{s_1}}{J(h)^2} + \left\{ g\cos\theta + \frac{\kappa\hat{u}^2}{J(h)^3} \right\} h_{s_1} - \left\{ \kappa g \sin\theta - \frac{\kappa_s \hat{u}^2}{J(h)^3} \right\} h + g\sin\theta = 0, \tag{1}$$

$$h_t + \frac{\hat{u}h_{s_1}}{J(h)^2} - \frac{\log\{J(h)\}}{J(h)\kappa}\hat{u}_{s_1} + \frac{\kappa_{s_1}}{\kappa}\left\{ \frac{\kappa h}{J(h)^2} + \frac{\log\{J(h)\}}{J(h)} \right\}\hat{u} = 0, \tag{2}$$

$$v(s_1, s_2, t) = \frac{\log\{J(s_2)\}\hat{u}_{s_1}}{J(s_2)\kappa} - \frac{\kappa_{s_1}}{\kappa^2}\left\{ \frac{\kappa s_2}{J(s_2)^2} + \frac{\log\{J(s_2)\}}{J(s_2)} \right\}\hat{u} = 0, \tag{3}$$

$$P(s_1, s_2, t) = \rho g \cos\theta \{h - s_2\} + \frac{\rho\hat{u}^2}{2}\{J(h)^{-2} - J(s_2)^{-2}\}, \tag{4}$$

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where alphabetic subscripts indicate differentiation; and t denotes the time, s_1 is the co-ordinate parallel to the flow bed, s_2 is the co-ordinate orthogonal to s_1, κ is the bed curvature, $\kappa(s_1)$; θ is the angle from horizontal to the tangent of the channel bed, $\theta(s_1)$; h is the depth, $h(s_1, t)$; and J is the Jacobian, $J=(1-\kappa(s_1)\ s_2)$, $J(s_1, s_2)$; u is the current velocity in the s_1 direction $u=\hat{u}/J(s_2)$, $u(s_1, s_2, t)$; $\hat{u}$ is the current velocity in the s_1 direction at the channel bottom, $\hat{u}=u(s_1, 0, t)$; v is the current velocity in the s_2 direction, $v(s_1, s_2, t)$; ρ is the density, assumed to be a constant; and g is the acceleration of gravity.

Dressler's expansion was applied to the two-dimensional Euler equations in the orthogonal curvilinear system defined by s_1, s_2. In the same manner as Friedrichs, irrotationality was assumed. The other basic assumptions were constant density, and an ambient surface pressure of zero. The usual kinematic surface conditions, i.e. no penetration at the channel bed, and a particle on the free surface remains on the surface, were enforced. The irrotationality assumption is reasonable for converging flow [5], as is the case in the vicinity of the spillway crest, and it has been used previously in this manner [6,7]. Even after the development of the turbulent boundary layer, this assumption for the flow profile is quite reasonable. The resistive action of the channel and eddy viscosities resulting from turbulence can be included in an empirical term such as the Chezy formula. Furthermore, Dressler [8] developed corrections to the Chezy and Manning coefficients to accommodate bed curvature.

Dressler's formulation yields u, h, and P to order ε^0 and v to order ε^1. The value of v to order ε^0 is simply $v\equiv 0$, which is identical to the result in the standard shallow water theory. However, given the zero-order perturbation for the solution u and h, the next higher approximation of v may be calculated. This treatment utilized the evaluation of the equations at the channel bed to remove some complicating terms. Sivakumaran [9,10] generalized the derivation further to a two-dimensional surface. Moreover, he placed these equations in a conservative form which is computationally appealing. Again, irrotationality was assumed. The one-dimensional equations, when evaluated at the bed, become

$$J(h)h_t+\frac{\partial}{\partial s_1}\left[-\frac{\hat{u}}{\kappa}\log[J(h)]\right]=0, \tag{5}$$

$$\hat{u}_t+gE_{s_1}=0, \tag{6}$$

where

$$E(s_1, t)\equiv \zeta+h\cos\theta+\frac{P_0}{\rho g}+\frac{1}{J(h)^2}\frac{\hat{u}^2}{2g}$$

$$\frac{P-P_0}{\rho g}=[h-s_2]\cos\theta+[J(h)^{-2}-J(s_2)^{-2}]\frac{\hat{u}^2}{2g}, \tag{7}$$

$$v=\frac{1}{J(s_2)}\frac{\partial}{\partial s_1}\left[\frac{\hat{u}}{\kappa}\log\{J(s_2)\}\right], \tag{8}$$

and ζ is the elevation of the bed above some reference level with P_0 the ambient pressure at the free surface. Both of these sets shall be referred to as Dressler's equations throughout the present study, as these two formulations are equivalent. This model appears to be applicable for $-0.85\leq\kappa h<0.5$ and Sivakumaran [9,11] has demonstrated a good agreement with experimental results over the wider range $-2\leq\kappa h<0.54$.

The free surface and the non-hydrostatic pressure aspects of the flow were included in the previous derivations, but vorticity about the bed-normal direction was not. The variation in pressure due to curvature may rival that of the hydrostatic pressure, and it is also important that eddy patterns and vorticity resulting from sidewalls and the drag due to bottom friction is reflected in the model to distinguish the design alternatives. These problems cannot be adequately modeled using the previous perturbation formulations. The present approach yields a more general formulation that includes the vorticity features.

2. EQUATION DEVELOPMENT

The derivation developed here employs concepts common to the studies of Friedrichs, Keller and Dressler and also involves easing of the irrotationality restriction with extension to a two-dimensional surface. The basic approach is to use an asymptotic expansion of the dependent variables of the three-dimensional Euler equations (written in an orthogonal curvilinear co-ordinate system) in a shallowness parameter ε.

The derivation begins with the Euler equations

$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

$$\mathbf{v}_t + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - \mathbf{v}\times\boldsymbol{\omega} + \frac{1}{\rho}\nabla P - \mathbf{g} = \mathbf{0}, \tag{10}$$

where, $\boldsymbol{\omega}$ is the vorticity vector, and $\mathbf{g}$ is the body force. The free-surface kinematic boundary condition requires that a particle on the surface remain on the surface, so that

$$h(s_1, s_2, t) - s_3 = 0. \tag{11}$$

This may be written

$$h_t + \mathbf{v}(s_1, s_2, h, t)\cdot\nabla h = w(s_1, s_2, h, t), \tag{12}$$

where, u, v, and w are in the s_1, s_2, and s_3 directions, respectively. There is also a bottom kinematic boundary condition, that the velocity normal to the bed is zero. This implies

$$w(s_1, s_2, 0, t) = 0. \tag{13}$$

The pressure at the free surface is a constant, in this case the reference value zero is used,

$$P(s_1, s_2, h, t) = 0. \tag{14}$$

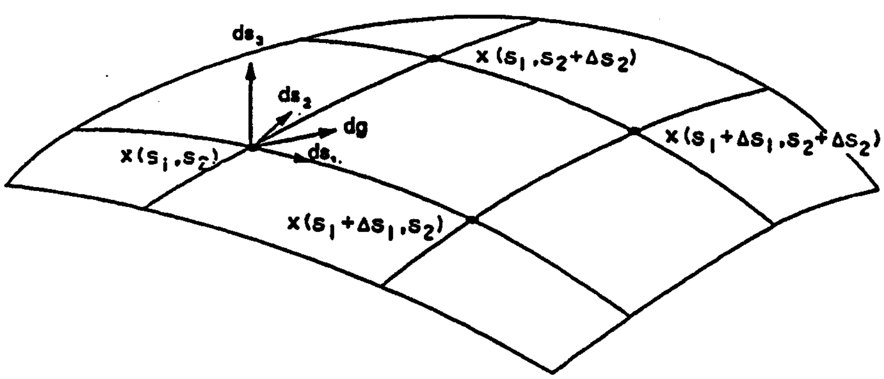

Figure 1. Co-ordinate system.

The irrotationality condition implies that the vorticity vector components in the s_1 and s_2 directions are zero,

$$\omega_1 = \omega_2 = 0. \tag{15}$$

The co-ordinate system shown in Figure 1 is a mutually orthogonal system. The co-ordinate directions s_1 and s_2 are in fact curvilinear, and s_3 is normal to the surface defined in this way. Therefore, $s_3(x, y, z) = c$ (where x, y, z are Cartesian co-ordinates) defines a co-ordinate surface above the bed. An infinitesimal vector **dg** that lies within the bed has a length given by

$$|\mathbf{dg}|^2 = \left(\frac{\partial \mathbf{x}}{\partial s_1}\,\mathrm{d}s_1 + \frac{\partial \mathbf{x}}{\partial s_2}\,\mathrm{d}s_2\right)\cdot\left(\frac{\partial \mathbf{x}}{\partial s_1}\,\mathrm{d}s_1 + \frac{\partial \mathbf{x}}{\partial s_2}\,\mathrm{d}s_2\right), \tag{16}$$

or

$$|\mathbf{dg}|^2 = \zeta_1^2(\mathrm{d}s_1)^2 + \zeta_2^2(\mathrm{d}s_2)^2, \tag{17}$$

where

$$\zeta_1 = \left(\frac{\partial \mathbf{x}}{\partial s_1}\cdot\frac{\partial \mathbf{x}}{\partial s_1}\right)^{1/2}, \qquad \zeta_2 = \left(\frac{\partial \mathbf{x}}{\partial s_2}\cdot\frac{\partial \mathbf{x}}{\partial s_2}\right)^{1/2}. \tag{18}$$

For the surface-normal co-ordinate $(\partial \mathbf{x}/\partial s_3 \cdot \partial \mathbf{x}/\partial s_3)^{1/2} \equiv 1$. Advancement in the s_3 direction shows that ζ_1 and ζ_2 are scaled in relation to s_3 as $j_1 \equiv \zeta_1(1 - \kappa_1 s_3)$, and similarly for the s_2 direction. This formulation of the metrics requires that s_1 and s_2 are not only orthogonal, but are also principal directions. The curvature in the s_1 direction, for example, is given by

$$\kappa_1 = -\frac{\partial \mathbf{x}/\partial s_1}{(\partial \mathbf{x}/\partial s_1 \cdot \partial \mathbf{x}/\partial s_1)^{1/2}}\cdot\frac{\partial \mathbf{N}/\partial s_1}{(\partial \mathbf{x}/\partial s_1 \cdot \partial \mathbf{x}/\partial s_1)^{1/2}}, \tag{19}$$

where **N** is the unit vector normal to the s_1–s_2 surface; if $\partial \mathbf{N}/\partial s_1$ is parallel to the s_1 tangent vector then the curvature along s_1 is an extremum and s_1 is a principal direction. The normal vector remains in the s_1–s_2 plane; s_2 is normal to s_1 and hence is also a principal direction. Introducing this co-ordinate system for (9)–(15) and simplifying, gives the following system

Continuity equation

$$\frac{\partial(j_2 u)}{\partial s_1} + \frac{\partial(j_1 v)}{\partial s_2} + \frac{\partial(j_1 j_2 w)}{\partial s_3} = 0. \tag{20}$$

s_1 momentum equation

$$u_t + \frac{u}{j_1}u_{s_1} + \frac{v}{j_2}u_{s_2} + wu_{s_3} = \frac{v^2}{j_1 j_2}\frac{\partial j_2}{\partial s_1} + \frac{uv}{j_1 j_2}\frac{\partial j_1}{\partial s_2} - \left(\frac{\zeta\kappa}{j}\right)_1 uw + \frac{1}{j_i\rho}P_{s_1} - g_1 = 0. \tag{21}$$

s_2 momentum equation

$$v_t + \frac{u}{j_1}v_{s_1} + \frac{v}{j_2}v_{s_2} + wv_{s_3} - \frac{u^2}{j_1 j_2}\frac{\partial j_1}{\partial s_2} + \frac{uv}{j_1 j_2}\frac{\partial j_2}{\partial s_1} - \left(\frac{\zeta\kappa}{j}\right)_2 vw + \frac{1}{j_2\rho}P_{s_2} - g_2 = 0. \tag{22}$$

s_3 momentum equation

$$w_t + \frac{u}{j_1}w_{s_1} + \frac{v}{j_2}w_{s_2} + ww_{s_3} + \left(\frac{\zeta\kappa}{j}\right)_1 u^2 + \left(\frac{\zeta\kappa}{j}\right)_2 v^2 + \frac{1}{\rho}P_{s_3} - g_3 = 0. \tag{23}$$

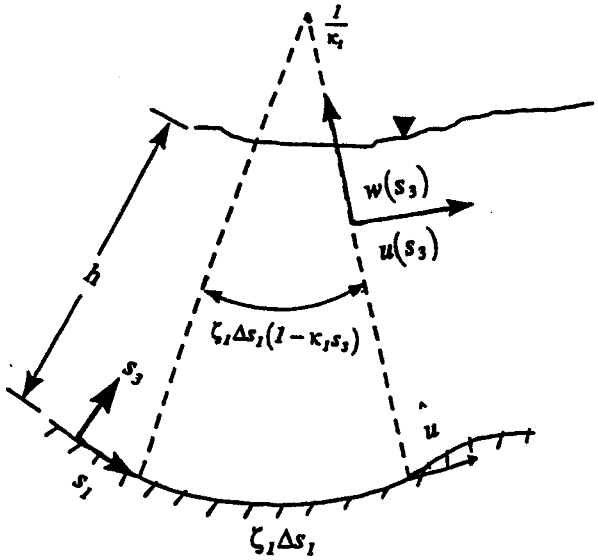

Figure 2. Equation variables in the s_1–s_3 plane.

Irrotationality condition

$$w_{s_2} - (j_2 v)_{s_3} = 0, \tag{24}$$

$$(j_1 u)_{s_3} - w_{s_1} = 0. \tag{25}$$

Free-surface boundary conditions (at $s_3 = h$)

$$h_t + \frac{u}{j_1} h_{s_1} + \frac{v}{j_2} h_{s_2} = w, \tag{26}$$

$$P(s_1, s_2, h, t) = 0. \tag{27}$$

Channel bed boundary condition

$$w(s_1, s_2, 0, t) = 0. \tag{28}$$

These equations are quite complex and the present objective is to extract the important simplifications of these equations that result when the flow depth is small. Figure 2 shows these variables in the s_1–s_3 plane, which is similar to the s_2–s_3 plane.

In the manner of Friedrichs, the equations may be non-dimensionalized and the dependent variables expanded in powers of ε. Typical length scales are the depth 'd' and the free-surface radius of curvature 'l'. As in shallow water theory, the relationship with velocity and thus time is assumed by the approximate celerity of a free-surface wave, $(gd)^{1/2}$ [1].

Non-dimensionalizing, let α, β, γ and τ be the new independent variables, so that

$$\begin{aligned} &s_1 = l\alpha \quad u = (gd)^{1/2}\tilde{u} \quad \zeta_1 = \tilde{\zeta}_1 \quad \kappa_1 = \tilde{\kappa}_1/d \quad j_1 = \zeta_1(1-\tilde{\kappa}_1\beta) \\ &s_2 = l\gamma \quad v = (gd)^{1/2}\tilde{v} \quad \zeta_2 = \tilde{\zeta}_2 \quad \kappa_2 = \tilde{\kappa}_2/d \quad j_2 = \zeta_2(1-\tilde{\kappa}_2\beta) \\ &s_3 = d\beta \quad w = \frac{l(gd)^{1/2}}{d}\tilde{w} \quad P = \rho g d\pi \quad t = \frac{l}{(gd)^{1/2}}\tau \quad \left(\frac{d}{l}\right)^2 \equiv \varepsilon \\ &h = dY \end{aligned} \tag{29}$$

where the tildes ($\sim$) indicate non-dimensional quantities. Recasting the governing equations in the dimensionless form and dropping the tildes for convenience, Equations (20)–(28) become

Continuity equation

$$\varepsilon\{(j_2u)_\alpha + (j_1v)_\gamma\} + (j_1j_2w)_\beta = 0. \tag{30}$$

α momentum equation

$$\varepsilon\left\{u_\tau + \frac{u}{j_1}u_\alpha + \frac{v}{j_2}u_\gamma - \frac{v^2}{j_1j_2}\frac{\partial j_2}{\partial\alpha} + \frac{uv}{j_1j_2}\frac{\partial j_1}{\partial\gamma} + \frac{\pi_\alpha}{j_1} - F\right\} + wu_\beta - \left(\frac{\zeta\kappa}{j}\right)_1 uw = 0. \tag{31}$$

γ momentum equation

$$\varepsilon\left\{v_\tau + \frac{u}{j_1}v_\alpha + \frac{v}{j_2}v_\gamma - \frac{u^2}{j_1j_2}\frac{\partial j_1}{\partial\gamma} + \frac{uv}{j_1j_2}\frac{\partial j_2}{\partial\alpha} + \frac{\pi_\gamma}{j_2} - G\right\} + wv_\beta - \left(\frac{\zeta\kappa}{j}\right)_2 vw = 0. \tag{32}$$

β momentum equation

$$\varepsilon\left\{w_\tau + \frac{u}{j_1}w_\alpha + \frac{v}{j_2}w_\gamma + \left(\frac{\zeta\kappa}{j}\right)_1 u^2 + \left(\frac{\zeta\kappa}{j}\right)_2 v^2 + \pi_\beta - H\right\} + ww_\beta = 0. \tag{33}$$

Irrotationality condition

$$w_\gamma - (j_2v)_\beta = 0, \tag{34}$$

$$(j_1u)_\beta - w_\alpha = 0. \tag{35}$$

Free-surface boundary conditions (at $\beta = Y$)

$$\varepsilon\left\{Y_\tau + \frac{u}{j_1}Y_\alpha + \frac{v}{j_2}Y_\gamma\right\} = w, \tag{36}$$

$$\pi(\alpha, \gamma, Y, \tau) = 0. \tag{37}$$

Channel bed boundary condition

$$w(\alpha, \gamma, 0, \tau) = 0. \tag{38}$$

3. PERTURBATION ANALYSIS

A regular perturbation analysis in ε is now developed for the flow equations in the curvilinear system. Each dependent variable is expanded in a power series in ε as follows

$$\begin{aligned}
u(\alpha, \gamma, \beta, \tau; \varepsilon) &= u^{(0)}(\alpha, \gamma, \beta, \tau) + \varepsilon u^{(1)}(\alpha, \gamma, \beta, \tau) + \cdots \\
v(\alpha, \gamma, \beta, \tau; \varepsilon) &= v^{(0)}(\alpha, \gamma, \beta, \tau) + \varepsilon v^{(1)}(\alpha, \gamma, \beta, \tau) + \cdots \\
w(\alpha, \gamma, \beta, \tau; \varepsilon) &= w^{(0)}(\alpha, \gamma, \beta, \tau) + \varepsilon w^{(1)}(\alpha, \gamma, \beta, \tau) + \cdots \\
\pi(\alpha, \gamma, \beta, \tau; \varepsilon) &= \pi^{(0)}(\alpha, \gamma, \beta, \tau) + \varepsilon \pi^{(1)}(\alpha, \gamma, \beta, \tau) + \cdots \\
Y(\alpha, \gamma, \beta, \tau; \varepsilon) &= Y^{(0)}(\alpha, \gamma, \beta, \tau) + \varepsilon Y^{(1)}(\alpha, \gamma, \beta, \tau) + \cdots
\end{aligned} \tag{39}$$

These expansions are substituted into Equations (30)–(38), yielding

$$\varepsilon\{(j_2(u^{(0)}+\varepsilon u^{(1)}+\cdots))_\alpha+(j_1(v^{(0)}+\varepsilon v^{(1)}+\cdots))_\gamma\}+(j_1 j_2(w^{(0)}+\varepsilon w^{(1)}+\cdots))_\beta=0, \tag{40}$$

$$\begin{aligned}&\varepsilon\bigg\{(u^{(0)}+\varepsilon u^{(1)}+\cdots)_\tau+\frac{1}{j_1}(u^{(0)}+\varepsilon u^{(1)}+\cdots)(u^{(0)}+\varepsilon u^{(1)}+\cdots)_\alpha\\&+\frac{1}{j_2}(v^{(0)}+\varepsilon v^{(1)}+\cdots)(u^{(0)}+\varepsilon u^{(1)}+\cdots)_\gamma-\frac{1}{j_1 j_2}(v^{(0)}+\varepsilon v^{(1)}+\cdots)^2\frac{\partial j_2}{\partial\alpha}\\&+\frac{1}{j_1 j_2}(u^{(0)}+\varepsilon u^{(1)}+\cdots)(v^{(0)}+\varepsilon v^{(1)}+\cdots)\frac{\partial j_1}{\partial\gamma}+\frac{1}{j_1}(\pi^{(0)}+\varepsilon\pi^{(1)}+\cdots)_\alpha-F\bigg\}\\&+(w^{(0)}+\varepsilon w^{(1)}+\cdots)(u^{(0)}+\varepsilon u^{(1)}+\cdots)_\beta-\left(\frac{\zeta\kappa}{j}\right)_1(u^{(0)}+\varepsilon u^{(1)}+\cdots)(w^{(0)}+\varepsilon w^{(1)}+\cdots)=0,\end{aligned} \tag{41}$$

$$\begin{aligned}&\varepsilon\bigg\{(v^{(0)}+\varepsilon v^{(1)}+\cdots)_\tau+\frac{1}{j_1}(u^{(0)}+\varepsilon u^{(1)}+\cdots)(v^{(0)}+\varepsilon v^{(1)}+\cdots)_\alpha\\&+\frac{1}{j_2}(v^{(0)}+\varepsilon v^{(1)}+\cdots)(v^{(0)}+\varepsilon v^{(1)}+\cdots)_\gamma-\frac{1}{j_1 j_2}(u^{(0)}+\varepsilon u^{(1)}+\cdots)^2\frac{\partial j_1}{\partial\gamma}\\&+\frac{1}{j_1 j_2}(u^{(0)}+\varepsilon u^{(1)}+\cdots)(v^{(0)}+\varepsilon v^{(1)}+\cdots)\frac{\partial j_2}{\partial\alpha}+\frac{1}{j_2}(\pi^{(0)}+\varepsilon\pi^{(1)}+\cdots)_\gamma-G\bigg\}\\&+(w^{(0)}+\varepsilon w^{(1)}+\cdots)(v^{(0)}+\varepsilon v^{(1)}+\cdots)_\beta-\left(\frac{\zeta\kappa}{j}\right)_2(v^{(0)}+\varepsilon v^{(1)}+\cdots)(w^{(0)}+\varepsilon w^{(1)}+\cdots)=0,\end{aligned} \tag{42}$$

$$\begin{aligned}&\varepsilon\bigg\{(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\tau+\frac{1}{j_1}(u^{(0)}+\varepsilon u^{(1)}+\cdots)(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\alpha\\&+\frac{1}{j_2}(v^{(0)}+\varepsilon v^{(1)}+\cdots)(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\gamma+\left(\frac{\zeta\kappa}{j}\right)_1(u^{(0)}+\varepsilon u^{(1)}+\cdots)^2\\&+\left(\frac{\zeta\kappa}{j}\right)_2(v^{(0)}+\varepsilon v^{(1)}+\cdots)^2+(\pi^{(0)}+\varepsilon\pi^{(1)}+\cdots)_\beta-H\bigg\}\\&+(w^{(0)}+\varepsilon w^{(1)}+\cdots)(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\beta=0,\end{aligned} \tag{43}$$

$$(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\gamma-(j_2(v^{(0)}+\varepsilon v^{(1)}+\cdots))_\beta=0, \tag{44}$$

$$(j_1(u^{(0)}+\varepsilon u^{(1)}+\cdots))_\beta-(w^{(0)}+\varepsilon w^{(1)}+\cdots)_\alpha=0, \tag{45}$$

$$\begin{aligned}&\varepsilon\bigg\{(Y^{(0)}+\varepsilon Y^{(1)}+\cdots)_\tau\\&+\frac{1}{j_1(\alpha,\gamma,Y,\tau)}(u^{(0)}(\alpha,\gamma,Y,\tau)+\varepsilon u^{(1)}(\alpha,\gamma,Y,\tau)+\cdots)(Y^{(0)}+\varepsilon Y^{(1)}+\cdots)_\alpha\\&+\frac{1}{j_2(\alpha,\gamma,Y,\tau)}(v^{(0)}(\alpha,\gamma,Y,\tau)+\varepsilon v^{(1)}(\alpha,\gamma,Y,\tau)+\cdots)(Y^{(0)}+\varepsilon Y^{(1)}+\cdots)_\gamma\bigg\}\\&=w^{(0)}(\alpha,\gamma,Y,\tau)+\varepsilon w^{(1)}(\alpha,\gamma,Y,\tau)+\cdots\end{aligned} \tag{46}$$

$$\pi^{(0)}(\alpha, \gamma, Y, \tau) + \varepsilon\pi^{(1)}(\alpha, \gamma, Y, \tau) + \cdots = 0, \tag{47}$$

$$w^{(0)}(\alpha, \gamma, 0, \tau) + \varepsilon w^{(1)}(\alpha, \gamma, 0, \tau) + \cdots = 0. \tag{48}$$

Equations are developed by collecting terms of identical orders in ε. Thus, these equations reflect the relative significance of the shallowness of the flow. Beginning with the lowest-order effect (ε^0), the continuity equation in conjunction with the channel bed boundary condition imply that

$$w^{(0)} \equiv 0. \tag{49}$$

The irrotationality conditions yield the relationship between the flow velocities $u^{(0)}$, $v^{(0)}$ and depth

$$v^{(0)} = \frac{(\hat{v}\zeta_2)}{j_2}, \tag{50}$$

$$u^{(0)} = \frac{(\hat{u}\zeta_1)}{j_1}, \tag{51}$$

where

$$\hat{v} \equiv v^{(0)}(\alpha, \gamma, 0, \tau), \tag{52}$$

$$\hat{u} \equiv u^{(0)}(\alpha, \gamma, 0, \tau). \tag{53}$$

Now consider the first-order perturbation effect. The following continuity equation contribution is obtained

$$(j_2 u^{(0)})_\alpha + (j_1 v^{(0)})_\gamma + (j_1 j_2 w^{(1)})_\beta = 0.$$

Note that this can then be integrated with respect to β over the depth, to yield

$$\begin{aligned} w^{(1)}(Y) &= \frac{1}{j_1(Y) j_2(Y)} \left[-\frac{\partial}{\partial\alpha} \left[\frac{\hat{u}\zeta_2}{\kappa_1} \left\{ \left(\frac{\kappa_2 - \kappa_1}{\kappa_1} \right) \log(f_1(Y)) + \kappa_2 Y^{(0)} \right\} \right] \right. \\ &\quad \left. - \frac{\partial}{\partial\gamma} \left[\frac{\hat{v}\zeta_1}{\kappa_2} \left\{ \left(\frac{\kappa_1 - \kappa_2}{\kappa_2} \right) \log(f_2(Y)) + \kappa_1 Y^{(0)} \right\} \right] + \hat{u} \frac{j_2(Y)}{f_1(Y)} Y^{(0)}_\alpha + \hat{v} \frac{j_1(Y)}{f_2(Y)} Y^{(0)}_\gamma \right], \end{aligned} \tag{54}$$

where

$$f_1(Y^{(0)}) \equiv 1 - \kappa_1 Y^{(0)},$$

$$f_2(Y^{(0)}) \equiv 1 - \kappa_2 Y^{(0)}.$$

Substituting the free-surface equation resulting from the terms of order ε, we obtain

$$\begin{aligned} & j_1(Y) j_2(Y) Y^{(0)}_\tau + \frac{\partial}{\partial\alpha} \left[\hat{u} \frac{\zeta_2}{\kappa_1} \left\{ \left(\frac{\kappa_2 - \kappa_1}{\kappa_1} \right) \log(f_1(Y^{(0)})) + \kappa_2 Y^{(0)} \right\} \right] \\ & + \frac{\partial}{\partial\gamma} \left[\hat{v} \frac{\zeta_1}{\kappa_2} \left\{ \left(\frac{\kappa_1 - \kappa_2}{\kappa_2} \right) \log(f_2(Y^{(0)})) + \kappa_1 Y^{(0)} \right\} \right] = 0. \end{aligned} \tag{55}$$

If the momentum equation for the normal direction is integrated with respect to β, the zero-order perturbation pressure contribution is obtained

$$\pi^{(0)}(\beta)=\frac{1}{2}\{\hat{u}^2((f_1(Y^{(0)}))^{-2}-(f_1(\beta))^{-2})+\hat{v}^2((f_2(Y^{(0)}))^{-2}-(f_2(\beta))^{-2})\}+H\{Y^{(0)}-\beta\}. \tag{56}$$

The terms within the first set of braces are the contribution to the pressure from the curvature 'centrifugal' effect and the second set is simply the hydrostatic component. The momentum equation in the α-direction can be simplified by the realization that $u_\beta^{(0)}-(\zeta\kappa/j)_1u^{(0)}=0$ due to Equation (51). This implies

$$u_\tau^{(0)}+\frac{u^{(0)}u_\alpha^{(0)}}{j_1}+\frac{v^{(0)}u_\gamma^{(0)}}{j_2}-\frac{(v^{(0)})^2}{j_1j_2}\frac{\partial j_2}{\partial\alpha}+\frac{u^{(0)}v^{(0)}}{j_1j_2}\frac{\partial j_1}{\partial\gamma}+\frac{\pi_\alpha^{(0)}}{j_1}-F=0, \tag{57}$$

and similarly for the γ momentum equation,

$$v_\tau^{(0)}+\frac{u^{(0)}v_\alpha^{(0)}}{j_1}+\frac{v^{(0)}v_\gamma^{(0)}}{j_2}-\frac{(u^{(0)})^2}{j_1j_2}\frac{\partial j_1}{\partial\gamma}+\frac{u^{(0)}v^{(0)}}{j_1j_2}\frac{\partial j_2}{\partial\alpha}+\frac{\pi_\gamma^{(0)}}{j_2}-G=0. \tag{58}$$

Finally, transforming these equations back to their dimensional form gives

$$j_1(h)j_2(h)h_t+\frac{\partial}{\partial s_1}\left[\hat{u}\frac{\zeta_2}{\kappa_1}\left\{\left(\frac{\kappa_2-\kappa_1}{\kappa_1}\right)\log(1-\kappa_1h)+\kappa_2h\right\}\right]$$
$$+\frac{\partial}{\partial s_2}\left[\hat{v}\frac{\zeta_1}{\kappa_2}\left\{\left(\frac{\kappa_1-\kappa_2}{\kappa_2}\right)\log(1-\kappa_2h)+\kappa_1h\right\}\right]=0, \tag{59}$$

$$u_t+\frac{u}{j_1}u_{s_1}+\frac{v}{j_2}u_{s_2}-\frac{v^2}{j_1j_2}\frac{\partial j_2}{\partial s_1}+\frac{uv}{j_1j_2}\frac{\partial j_1}{\partial s_2}+\frac{1}{j_1\rho}P_{s_1}-g_1=0, \tag{60}$$

$$v_t+\frac{u}{j_1}v_{s_1}+\frac{v}{j_2}v_{s_2}-\frac{u^2}{j_1j_2}\frac{\partial j_1}{\partial s_2}+\frac{uv}{j_1j_2}\frac{\partial j_2}{\partial s_1}+\frac{1}{j_2\rho}P_{s_2}-g_2=0, \tag{61}$$

$$w(s_3)=-\frac{1}{j_1j_2}\left[\frac{\partial}{\partial s_1}\left\{\frac{\hat{u}\zeta_2}{\kappa_1}\left(\left(\frac{\kappa_2-\kappa_1}{\kappa_1}\right)\log(1-\kappa_1s_3)+\kappa_2s_3\right)\right\}\right.$$
$$\left.+\frac{\partial}{\partial s_2}\left\{\frac{\hat{v}\zeta_1}{\kappa_2}\left(\left(\frac{\kappa_1-\kappa_2}{\kappa_2}\right)\log(1-\kappa_2s_3)+\kappa_1s_3\right)\right\}\right], \tag{62}$$

$$P(s_3)=\frac{\rho}{2}\left[\hat{u}^2\left\{\left(\frac{1}{1-\kappa_1h}\right)^2-\left(\frac{1}{1-\kappa_1s_3}\right)^2\right\}+\hat{v}^2\left\{\left(\frac{1}{1-\kappa_2h}\right)^2-\left(\frac{1}{1-\kappa_2s_3}\right)^2\right\}\right]+\rho g_3(h-s_3), \tag{63}$$

$$u(s_3)=\frac{\hat{u}}{1-\kappa_1s_3}, \tag{64}$$

$$v(s_3)=\frac{\hat{v}}{1-\kappa_2s_3}. \tag{65}$$

This system of equations can be used to solve for h, u, v and P and, from this information, w. A finite element model based on this system is described in Part II and numerical studies are presented.

4. CONCLUSIONS

A new formulation of the shallow water equations in a bed-fitted co-ordinate system has been developed. The formulation includes bed curvature effects and, therefore, is non-hydrostatic.

Furthermore, it does not restrict the flow field to be fully irrotational and allows vorticity components which have axes normal to the bed. Previous models that assume irrotationality and/or no curvature effects can be deduced directly from the more general form presented here. The model may be used for flow computations in applications such as spillways, as seen in Part II following where a finite element formulation and flume comparison studies are presented.

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